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CHARACTERISTIC FUNCTIONS OF LIFTINGS–II

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Abstract. We prove that the symbol of the characteristic function of a minimal contractive lifting is an injective map and that the converse also holds, using explicit computation and functional models. We discuss when the characteristic function of a lifting is a polynomial and present a series representation for the characteristic functions of liftings.

1. Introduction

Given a Hilbert space $\mathscr{L}$, a d-tuple $\underline{T} = (T_1, \ldots, T_d)$, such that $T_j \in \mathscr{B}(\mathscr{L})$ for $j = 1, \ldots, d$, is said to be a *row contraction* if $\sum_{j=1}^{d} T_j T_j^* \leqslant I_{\mathscr{L}}$. A row contraction $\underline{E} = (E_1, \ldots, E_d)$ on Hilbert space $\mathscr{H}_E$ is said to be a *contractive lifting* of a row contraction $\underline{C} = (C_1, \ldots, C_d)$ on a Hilbert space $\mathscr{H}_C$ if $\mathscr{H}_C \subset \mathscr{H}_E$ and $E_i^* h_C = C_i^* h_C$ for $h_C \in \mathscr{H}_C, i = 1, \ldots, d$. The contractive lifting $\underline{E}$ is said to be *minimal* if $\mathscr{H}_E$ is the smallest $\underline{E}$-invariant subspace containing $\mathscr{H}_C$. The theory of characteristic functions of row contractions was first studied by G. Popescu in [11]. Motivated by the theory of characteristic functions of row contractions and by applications to dynamics of open quantum systems (see [7], [8]), the notion of characteristic functions for liftings of row contractions was introduced in [3] which completely classify up to unitary equivalence certain class of liftings called reduced liftings. The notion of reduced lifting has been shown to be same as the notion of minimal contractive lifting in [4]. The current article is a sequel to our article [3].

The full *Fock space* over $\mathbb{C}^d$, denoted by Γ, is the Hilbert space

$$\Gamma := \mathbb{C} \oplus \mathbb{C}^d \oplus (\mathbb{C}^d)^{\otimes^2} \oplus \ldots \oplus (\mathbb{C}^d)^{\otimes^m} \oplus \ldots.$$

Fock spaces are very useful for constructing functional models for row contractions. The element $e_\emptyset := 1 \oplus 0 \oplus \ldots$ of Γ is called the vacuum vector. Let $\{e_1, \ldots, e_d\}$ be the standard orthonormal basis of $\mathbb{C}^d$. Let $L_j : \Gamma \to \Gamma$ denotes the left creation operator

$$L_j x = e_j \otimes x \text{ for all } x \in \Gamma, \quad j = 1, \ldots, d.$$

Our notation also include the case $d = \infty$ here and in this case $\mathbb{C}^d$ stands for a complex separable Hilbert space of infinite dimension.

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Suppose $\tilde{\Lambda}$ is the unital free semi-group with generators $1,\ldots,d$ and the identity $\emptyset$. If $T_j \in \mathscr{B}(\mathscr{L})$ for $j = 1,\ldots,d$, then for $\alpha \in \tilde{\Lambda}$ define

$$T_\alpha := \begin{cases} T_{\alpha_1} \ldots T_{\alpha_m} & \text{if } \alpha = \alpha_1 \ldots \alpha_m, \quad \alpha_i \in \{1,\ldots,d\} \\ I_{\mathscr{L}} & \text{if } \alpha = \emptyset. \end{cases}$$

For $\alpha = \alpha_1 \ldots \alpha_m \in \tilde{\Lambda}$ we denote the vector $e_{\alpha_1} \otimes \ldots \otimes e_{\alpha_m}$ by e_α in the full Fock space Γ. Then $\{e_\alpha : \alpha \in \tilde{\Lambda}\}$ forms an orthonormal basis for the full Fock space Γ. The length of $\alpha \in \tilde{\Lambda}$ is defined to be m if $\alpha = \alpha_1 \ldots \alpha_m$, and 0 if $\alpha = \emptyset$. It is denoted by $|\alpha|$.

DEFINITION 1. Let $\mathscr{E}$ and $\mathscr{E}_*$ be Hilbert spaces. A bounded operator $M : \Gamma \otimes \mathscr{E} \to \Gamma \otimes \mathscr{E}_*$ is said to be a *multi-analytic operator* if

$$M(L_j \otimes I_{\mathscr{E}}) = (L_j \otimes I_{\mathscr{E}_*})M \quad \text{for} \quad j = 1,\ldots,d. \tag{1}$$

Characteristic functions of row contractions as well as characteristic functions of liftings of row contractions are contractive multi-analytic operators. Contractive multi-analytic operators are noncommutative analogues of operator valued Schur class functions. Let $M : \Gamma \otimes \mathscr{E} \to \Gamma \otimes \mathscr{E}_*$ be a multi-analytic operator. The map defined by $\Theta := M|_{e_\emptyset \otimes \mathscr{E}} : e_\emptyset \otimes \mathscr{E} \to \Gamma \otimes \mathscr{E}_*$ is called the *symbol* of M and it uniquely determines M, i.e., if we define the multi-analytic operator $M_\Theta : \Gamma \otimes \mathscr{E} \to \Gamma \otimes \mathscr{E}_*$ by

$$M_\Theta(L_\alpha \otimes I_{\mathscr{E}})(e_\emptyset \otimes \ell) := (L_\alpha \otimes I_{\mathscr{E}_*})\Theta(e_\emptyset \otimes \ell)$$

for all $\ell \in \mathscr{E}, \alpha \in \tilde{\Lambda}$, then $M_\Theta = M$.

In Section 2 we show that the symbol of any characteristic function of a minimal contractive lifting is injective. We develop a functional model for contractive multi-analytic operators (cf. [4]). A complete answer is given for the following question in Section 3: If $\underline{C} = (C_1,\ldots,C_d)$ is a row contraction on a Hilbert space $\mathscr{H}_C$ and $\tilde{M} : \Gamma \otimes \mathscr{D} \to \Gamma \otimes \mathscr{D}_C$ is a contractive multi-analytic operator where $\mathscr{D}$ is any Hilbert space, then which intrinsic properties of a characteristic function of a lifting when assumed for $\tilde{M}$, guarantee that $\tilde{M}$ is the characteristic function for a minimal contractive lifting of $\underline{C}$? For every row contraction $\underline{C}$ and contractive multi-analytic operator whose codomain is $\Gamma \otimes \mathscr{D}_C$, an associated lifting of $\underline{C}$ was defined in [3]. It is shown here that if this contractive multi-analytic operator has injective symbol, then the multi-analytic operator can be realized as the characteristic function of the associated lifting. These results were proved previously in [4] using different methods and our approach here makes use of explicit computation and functional models. An example is worked out in Section 4 where we illustrate the constructions from the Section 3, viz., for a given row contraction and a certain contractive multi-analytic operator, the associated lifting of the given contraction is obtained and the characteristic function of the lifting is compared with the given multi-analytic operator.

The last section contains a criterion for the characteristic function of lifting to be a polynomial. Such studies were done for characteristic functions of contractions in [6]. G. Popescu carried out similar investigation in [12] for characteristic functions of row contractions and developed notions to detect when the characteristic function of any row contraction is a polynomial. Later in the last section, we derive a transfer function type series representation for the characteristic function of any contractive lifting. In [5] and [9], outgoing Cuntz Scattering systems were associated to coisometric liftings of row contractions and the characteristic functions of these liftings were shown to coincide with transfer functions of the scattering systems.

If a row contraction consists of isometries with orthogonal ranges, then it is called a *row isometry*. For a row contraction $\underline{T}$, a minimal contractive lifting is called a *minimal isometric dilation*, if the lifting is a row isometry. Let $\underline{T}$ be a row contraction on a Hilbert space $\mathscr{L}$. Define defect operators $D_T = (I - \underline{T}^*\underline{T})^{\frac{1}{2}} : \oplus_{i=1}^d \mathscr{L} \to \oplus_{i=1}^d \mathscr{L}$ and $D_{*,T} := (I - \underline{T}\underline{T}^*)^{\frac{1}{2}} : \mathscr{L} \to \mathscr{L}$. Denote $\mathscr{D}_T := \overline{\text{Range } D_T}$ and $\mathscr{D}_{*,T} := \overline{\text{Range } D_{*,T}}$. A construction of the minimal isometric dilation $\underline{V}$ (cf. [10]) of $\underline{T}$ on $\hat{\mathscr{L}} = \mathscr{L} \oplus (\Gamma \otimes \mathscr{D}_T)$ is

$$V_j(\ell \oplus \sum_{\alpha \in \tilde{\Lambda}} e_\alpha \otimes d_\alpha) = T_j \ell \oplus [e_\emptyset \otimes (D_T)_j \ell + e_j \otimes \sum_{\alpha \in \tilde{\Lambda}} e_\alpha \otimes d_\alpha] \tag{2}$$

where $\ell \in \mathscr{L}$, $d_\alpha \in \mathscr{D}_T$, and $(D_T)_j : \mathscr{L} \to \bigoplus_1^d \mathscr{L}$ is defined for $j = 1, \ldots, d$ by the $(D_T)_j \ell = D_T(0, \ldots, \ell, \ldots, 0)$ with ℓ is embedded at the j^{th} component. This construction is vital for the analysis done in this article.

2. Properties of the characteristic functions of liftings

In this section we discuss some important properties of the characteristic function of a minimal contractive lifting. Let $\underline{C} = (C_1, \ldots, C_d)$ be a row contraction on a Hilbert space $\mathscr{H}_C$ and $\underline{E} = (E_1, \ldots, E_d)$ be a lifting of $\underline{C}$ on a Hilbert space $\mathscr{H}_E \supset \mathscr{H}_C$. Let

$$\begin{pmatrix} C_j & 0 \\ B_j & A_j \end{pmatrix}$$

be the block matrix representation of E_j for $j = 1, \ldots, d$ with respect to $\mathscr{H}_C \oplus \mathscr{H}_C^\perp$. Now onward we denote $\mathscr{H}_C^\perp$ by $\mathscr{H}_A$. Let $\underline{V}^E = (V_1^E, \ldots, V_d^E)$ be a minimal isometric dilation of $\underline{E}$ on the Hilbert space $\mathscr{H}_E \oplus (\Gamma \otimes \mathscr{D}_E)$. Since $\underline{V}^E$ is a row isometry and $(V_j^E)^* h_C = E_j{}^* h_C = C_j{}^* h_C$ for all $h_C \in \mathscr{H}_C, j = 1, \ldots, d$, $\underline{V}^E$ is an isometric lifting of $\underline{C}$. The subspace $\tilde{\mathscr{H}}_C := \overline{\text{span}}\{V_\alpha^E h_C : h_C \in \mathscr{H}_C, \alpha \in \tilde{\Lambda}\}$ of $\mathscr{H}_E \oplus (\Gamma \otimes \mathscr{D}_E)$ is a reducing subspace for $\underline{V}^E$ because $(V_i^E)^* V_j^E = \delta_{ij} I$ for $i, j = 1, \ldots, d$. Since any minimal isometric dilation of a row contraction is unique upto unitary equivalence and the row isometry $V^E|_{\tilde{\mathscr{H}}_C}$ is a minimal isometric dilation of $\underline{C}$, we can embed the space $\mathscr{H}_C \oplus (\Gamma \otimes \mathscr{D}_C)$ of the minimal isometric dilation $\underline{V}^C = (V_1^C, \ldots, V_d^C)$ of $\underline{C}$ in $\mathscr{H}_E \oplus (\Gamma \otimes \mathscr{D}_E)$ as a reducing subspace of V_i^E's. In other words, there exist a row isometry $\underline{Y} = (Y_1, \ldots, Y_d)$ on a space $\mathscr{K}$ and a unitary $W : \mathscr{H}_E \oplus (\Gamma \otimes \mathscr{D}_E) \to \mathscr{H}_C \oplus (\Gamma \otimes \mathscr{D}_C) \oplus \mathscr{K}$ such that with $\tilde{V}_j^E := V_j^C \oplus Y_j$ we have

$$\tilde{V}_j^E W = W V_j^E \text{ and } W|_{\mathscr{H}_C} = I|_{\mathscr{H}_C}$$

for $j = 1, \ldots, d$. The *characteristic function* for the lifting $\underline{E}$ of $\underline{C}$ is introduced in [3] as the multi-analytic operator

$$M_{C,E} := P_{\Gamma \otimes \mathscr{D}_C} W|_{\Gamma \otimes \mathscr{D}_E} : \Gamma \otimes \mathscr{D}_E \to \Gamma \otimes \mathscr{D}_C$$

A row contraction $\underline{T} = (T_1, \ldots, T_d)$ on a Hilbert space $\mathscr{L}$ is said to be

(1) a *completely non-coisometric* (c.n.c.) tuple if

$$\{\ell \in \mathscr{L} : \sum_{|\alpha|=n} \|(T_\alpha)^* \ell\|^2 = \|\ell\|^2 \text{ for all } n \in \mathbb{N}\} = \{0\}.$$

(2) a *pure* or $*$-*stable* tuple if $\lim_{n\to\infty} \sum_{|\alpha|=n} \|(T_\alpha)^* \ell\|^2 = 0$ for all $\ell \in \mathscr{L}$.

It is easy to verify that if a row contraction $\underline{T} = (T_1, \ldots, T_d)$ is pure, then it is c.n.c.

We recall from Proposition 3.1 of [3] that if $\underline{E}$ is a contractive lifting of $\underline{C}$ by $\underline{A}$, then there exists a contraction $\gamma : \mathscr{D}_{*,A} \to \mathscr{D}_C$ such that $B_j^* = (D_C)_j^* \gamma D_{*,A}$ holds for $j = 1, \ldots, d$. The characteristic function $M_{C,E}$ of the lifting $\underline{E}$ has the following series expansion for its symbol $\Theta_{C,E}$: For $h_C \in \mathscr{H}_C$

$$\Theta_{C,E}(D_E)_j h_C = e_\emptyset \otimes [(D_C)_j h_C - \gamma D_{*,A} B_j h_C] - \sum_{|\alpha| \geqslant 1} e_\alpha \otimes \gamma D_{*,A} (A_\alpha)^* B_j h_C, \quad (3)$$

and for $h_A \in \mathscr{H}_A$

$$\Theta_{C,E}(D_E)_j h_A = -e_\emptyset \otimes \gamma D_{*,A} A_j h_A + \sum_{i=1}^{d} e_i \otimes \sum_{\alpha \in \tilde{\Lambda}} e_\alpha \otimes \gamma D_{*,A} (A_\alpha)^* (\delta_{ij} I - A_i^* A_j) h_A \quad (4)$$

where $j = 1, \ldots, d$.

For a contractive lifting $\underline{E}$ of $\underline{C}$, we have constructed the minimal isometric dilation $\underline{V}^E$ on the Hilbert space $\mathscr{H}_E \oplus (\Gamma \otimes \mathscr{D}_E)$ in the introduction. Because $\underline{V}^E$ is an isometric lifting of $\underline{C}$ on $\mathscr{H}_C$, we obtain the minimal isometric dilation $\underline{V}^C$ by restricting $\underline{V}^E$ to $\mathscr{H}_C \oplus \bigoplus_\alpha V_\alpha^E \mathscr{L}_C$, where $\mathscr{L}_C = \overline{\text{span}}\{\mathscr{H}_C, \underline{V}^E(\bigoplus_1^d \mathscr{H}_C)\} \ominus \mathscr{H}_C$ is a subspace with dimension equal to the dimension of the defect space $\mathscr{D}_C$ of $\underline{C}$ and hence can be identified canonically with $\mathscr{D}_C$. We restate below Proposition 3.8 of [4] in a form appropriate to the study done in the article:

THEOREM 1. A contractive lifting $\underline{E}$ of $\underline{C}$ by $\underline{A}$ is a minimal contractive lifting if and only if $\underline{A}$ is c.n.c. and the contraction $\gamma : \mathscr{D}_{*,A} \to \mathscr{D}_C$ is *resolving*, i.e., for $h_A \in \mathscr{H}_A$ we have

$$(\gamma D_{*,A} A_\alpha^* h_A = 0 \text{ for all } \alpha \in \tilde{\Lambda}) \Rightarrow (D_{*,A} A_\alpha^* h_A = 0 \text{ for all } \alpha \in \tilde{\Lambda}).$$

Proof. Let $h_E \in \mathcal{H}_E$. We have

$$\begin{aligned}
&h_E \in \mathcal{H}_E \ominus \left[\overline{\operatorname{span}}\{E_\alpha h_C\colon h_C \in \mathcal{H}_C,\ \text{all words } \alpha\}\right] \\
\Leftrightarrow\ & h_E \perp P_{\mathcal{H}_E} V_\alpha^E h_C \text{ for all } h_C \in \mathcal{H}_C,\ \text{all words } \alpha \\
\Leftrightarrow\ & h_E \perp P_{\mathcal{H}_E}\left[\mathcal{H}_C \oplus (\bigoplus_\alpha V_\alpha^E \mathcal{L}_C)\right] = \mathcal{H}_C + P_{\mathcal{H}_E} \bigoplus_\alpha V_\alpha^E \mathcal{L}_C \\
\Leftrightarrow\ & h_E \in \mathcal{H}_A \cap (\bigoplus_\alpha V_\alpha^E \mathcal{L}_C)^\perp
\end{aligned}$$

Therefore $\mathcal{H}_E \ominus \left[\overline{\operatorname{span}}\{E_\alpha h_C\colon h_C \in \mathcal{H}_C,\ \text{all words } \alpha\}\right] = \{0\}$ if and only if $\mathcal{H}_A \cap (\bigoplus_\alpha V_\alpha^E \mathcal{L}_C)^\perp = \{0\}$. It is immediate that the space $\mathcal{K}$, introduced in the first paragraph of this section, is embedded in the space $\mathcal{H}_E \oplus \Gamma \otimes \mathcal{D}_E$ by the unitary W^* as $(\bigoplus_\alpha V_\alpha^E \mathcal{L}_C)^\perp \ominus \mathcal{H}_C$. Thus, the assertion of the proposition follows from part $(i) \Longrightarrow (iii)$ of Lemma 3.5 of [3]. $\square$

We have remarked in the introduction that the notion of 'minimal contractive lifting' is same as the notion of 'reduced lifting' from [3]. Let us assume that $\underline{E}$ is a minimal contractive lifting of $\underline{C}$. Therefore it follows from Lemma 3.3 (iv) of [3] that

$$(\Gamma \otimes \mathcal{D}_C) \vee W(\Gamma \otimes \mathcal{D}_E) = (\Gamma \otimes \mathcal{D}_C) \oplus \mathcal{K}. \tag{5}$$

Define $\Delta_{C,E} := (I - M_{C,E}^* M_{C,E})^{1/2} : \Gamma \otimes \mathcal{D}_E \to \Gamma \otimes \mathcal{D}_E$. Then $x \in \Gamma \otimes \mathcal{D}_E$ we have

$$\begin{aligned}
\|P_{\mathcal{K}} Wx\|^2 &= \|(I - P_{\Gamma\otimes\mathcal{D}_C})Wx\|^2 = \|x\|^2 - \|P_{\Gamma\otimes\mathcal{D}_C} Wx\|^2 \\
&= \|x\|^2 - \|M_{C,E}x\|^2 = \|\Delta_{C,E}x\|^2.
\end{aligned} \tag{6}$$

Therefore there is a unitary operator $\Phi_{\mathcal{K}}$ from $\mathcal{K}$ onto $\overline{\Delta_{C,E}(\Gamma \otimes \mathcal{D}_E)}$ defined by

$$\Phi_{\mathcal{K}}(P_{\mathcal{K}} Wx) = \Delta_{C,E}x \text{ for } x \in \Gamma \otimes \mathcal{D}_E.$$

Since

$$W\mathcal{H}_A = [(\Gamma \otimes \mathcal{D}_C) \oplus \mathcal{K}] \ominus W(\Gamma \otimes \mathcal{D}_E), \tag{7}$$

with the unitary $\Phi := I \oplus \Phi_{\mathcal{K}} : (\Gamma \otimes \mathcal{D}_C) \oplus \mathcal{K} \to (\Gamma \otimes \mathcal{D}_C) \oplus \overline{\Delta_{C,E}(\Gamma \otimes \mathcal{D}_E)}$ we finally obtain for $x \in \Gamma \otimes \mathcal{D}_E$

$$\Phi W x = P_{\Gamma\otimes\mathcal{D}_C} Wx \oplus \Phi_{\mathcal{K}} P_{\mathcal{K}} Wx = M_{C,E}x \oplus \Delta_{C,E}x$$

and

$$\Phi W \mathcal{H}_A = [(\Gamma \otimes \mathcal{D}_C) \oplus \overline{\Delta_{C,E}(\Gamma \otimes \mathcal{D}_E)}] \ominus \{M_{C,E}x \oplus \Delta_{C,E}x : x \in \Gamma \otimes \mathcal{D}_E\}.$$

From these observations, a functional model is derived in Section 3 of [3] for the lifting $\underline{E}$ of $\underline{C}$.

DEFINITION 2. A multi-analytic operator $M_\Theta : \Gamma \otimes \mathcal{E} \to \Gamma \otimes \mathcal{E}_*$ is called *purely contractive* if its symbol Θ satisfies

$$\|P_{e_\emptyset \otimes \mathcal{E}_*} \Theta(\ell)\| < \|\ell\|$$

for all $0 \neq \ell \in \mathcal{E}$.

The characteristic function of a row contraction is purely contractive. But we observe for the following minimal contractive lifting that the characteristic function of the lifting is not purely contractive:

EXAMPLE. Let $C := \begin{pmatrix} \frac{\sqrt{3}}{2} & 0 \\ 0 & \frac{\sqrt{3}}{2} \end{pmatrix}$ on $\mathscr{H}_C = \mathbb{C}^2$ and let $E = \begin{pmatrix} C & 0 \\ B & A \end{pmatrix}$ on $\mathscr{H}_E = \mathscr{H}_C \oplus \mathscr{H}_A$ be a lifting of $\underline{C}$. Let $A := M_z$ on $\mathscr{H}_A = H^2$, the Hardy space, and $B : \mathscr{H}_C \to \mathscr{H}_A$ be defined by $B(x,y)^T = f_y$ where the superscript 'T' stands for transpose and $f_y(z) := \frac{1}{2}y$.

It follows that $D_C = (I - C^*C)^{\frac{1}{2}} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ and $D_{*,A} = (I - AA^*)^{\frac{1}{2}}$ is the orthogonal projection onto the space of constant functions inside H^2. Here $B^* : \mathscr{H}_A \to \mathscr{H}_C$ is given by $B^* f = (0, \frac{1}{2} f_0)^T$ where f_0 is the constant term of $f \in H^2$. Define an isometry $\gamma : \mathscr{D}_{*,A} \to \mathscr{D}_C$ by

$$\gamma\alpha := (0, \alpha)^T \text{ for } \alpha \in \mathscr{H}_C.$$

Observe that $B^* = D_C \gamma D_{*,A}$. Thus by Proposition 3.1 of [3] we infer that E is a contraction. Because A is the unilateral shift, it is is *pure*, i.e., for all $h_A \in \mathscr{H}_A$ we have $\lim_{n\to\infty} \|(A^*)^n h_A\| = 0$ and hence it is c.n.c. Since γ is an isometry, γ is resolving. From Theorem 2.1 we deduce that E is a minimal contractive lifting of C.

Next we show that the characteristic function $M_{C,E}$ for the lifting E of C is not purely contractive. Observe that $\ker B = \{(x,0)^T : x \in \mathbb{C}\}$. Suppose $h_C = (x_0, 0)^T \in \mathscr{H}_C$ for some $x_0 \neq 0$. Then

$$\begin{aligned} \|D_E h_C\|^2 &= \|h_C\|^2 - \|E h_C\|^2 = \|h_C\|^2 - [\|C h_C\|^2 + \|B h_C\|^2] \\ &= \|h_C\|^2 - \|C h_C\|^2 = \|D_C h_C\|^2 = \frac{1}{4}|x_0|^2 \neq 0. \end{aligned}$$

By equation (3) we have $\Theta_{C,E} D_E h_C = D_C h_C$ and so $\|P_{e_\emptyset \otimes \mathscr{D}_C} \Theta_{C,E} D_E h_C\|^2 = \|D_C h_C\|^2 = \|D_E h_C\|^2$. Hence $M_{C,E}$ is not purely contractive.

In the following proposition, we observe that the defect space $\mathscr{D}_E$ of any minimal contractive lifting $\underline{E}$ of a row contraction $\underline{C}$ is the span closure of certain subspaces. This will be vital in proving that the characteristic functions of such liftings are always injective.

PROPOSITION 1. *If $M_{C,E} : \Gamma \otimes \mathscr{D}_E \to \Gamma \otimes \mathscr{D}_C$ is the characteristic function for the minimal contractive lifting $\underline{E}$ of $\underline{C}$, then*

$$\begin{aligned} e_\emptyset \otimes \mathscr{D}_E = M^*_{C,E}(e_\emptyset \otimes \mathscr{D}_C) \vee \{ & M^*_{C,E}(L_j \otimes I)u + \lim_{n\to\infty} \Delta^2_{C,E}(L_j \otimes I)v_n : u \in \Gamma \otimes \mathscr{D}_C, \\ & v_n \in \Gamma \otimes \mathscr{D}_E, u \oplus \lim_{n\to\infty} \Delta_{C,E} v_n \perp M_{C,E}x \oplus \Delta_{C,E}x \text{ for all } x \in \Gamma \otimes \mathscr{D}_E,\ j = 1,\dots,d \}. \end{aligned}$$

Proof. We denote by $\mathscr{N}$ the space

$$M_{C,E}^*(e_\emptyset \otimes \mathscr{D}_C) \vee \{M_{C,E}^*(L_j \otimes I)u + \lim_{n\to\infty} \Delta_{C,E}^2(L_j \otimes I)v_n : \\ u \oplus \lim_{n\to\infty} \Delta_{C,E} v_n \perp M_{C,E}x \oplus \Delta_{C,E}x \text{ for all } x \in \Gamma \otimes \mathscr{D}_E,\ j = 1,\dots,d\}.$$

First note that for all $j = 1,\dots,d$ with $|\alpha| \geqslant 1$, and $d_C \in \mathscr{D}_C$ we have

$$(L_j^* \otimes I)M_{C,E}^*(e_\emptyset \otimes d_C) = M_{C,E}^*(L_j^* \otimes I)(e_\emptyset \otimes d_C) = 0.$$

This implies that $M_{C,E}^*(e_\emptyset \otimes d_C) \in e_\emptyset \otimes \mathscr{D}_E$. Let $h_E = h_C \oplus h_A \in \mathscr{H}_C \oplus \mathscr{H}_A (= \mathscr{H}_E)$ and $h \in \mathscr{H}_C$. For $1 \leqslant i,j \leqslant d$ we observe that

$$\begin{aligned} &\langle M_{C,E}^*(e_\emptyset \otimes (D_C)_j h), e_\emptyset \otimes (D_E)_i h_E\rangle \\ &= \langle e_\emptyset \otimes (D_C)_j h, M_{C,E}\big(e_\emptyset \otimes (D_E)_i (h_C \oplus h_A)\big)\rangle \\ &= \langle e_\emptyset \otimes (D_C)_j h, e_\emptyset \otimes [(D_C)_i h_C - \gamma D_{*,A} B_i h_C - \gamma D_{*,A} A_i h_A]\rangle \\ &= \langle e_\emptyset \otimes h, e_\emptyset \otimes (D_C)_j^*[(D_C)_i h_C - \gamma D_{*,A} B_i h_C - \gamma D_{*,A} A_i h_A]\rangle \\ &= \langle e_\emptyset \otimes h, e_\emptyset \otimes [(\delta_{ij} I - C_j^* C_i) h_C - B_j^* B_i h_C - B_j^* A_i h_A]\rangle \\ &= \langle e_\emptyset \otimes h, e_\emptyset \otimes (D_E)_j^* (D_E)_i h_E\rangle \\ &= \langle e_\emptyset \otimes (D_E)_j h, e_\emptyset \otimes (D_E)_i h_E\rangle. \end{aligned}$$

Thus $M_{C,E}^*(e_\emptyset \otimes (D_C)_j h) = e_\emptyset \otimes (D_E)_j h$ for all $h \in \mathscr{H}_C$, $j = 1,\dots,d$.

Since $\tilde{V}_j^E$ reduces $\mathscr{K}$, the projection $P_{\mathscr{K}}$ commutes with $\tilde{V}_j^E$ for $j = 1,\dots,d$. For $x \in \Gamma \otimes \mathscr{D}_E$ we have

$$\begin{aligned} \Phi_{\mathscr{K}} Y_j P_{\mathscr{K}} W x &= \Phi_{\mathscr{K}} \tilde{V}_j^E P_{\mathscr{K}} W x = \Phi_{\mathscr{K}} P_{\mathscr{K}} \tilde{V}_j^E W x \\ &= \Phi_{\mathscr{K}} P_{\mathscr{K}} W V_j^E x = \Phi_{\mathscr{K}} P_{\mathscr{K}} W (L_j \otimes I) x = \Delta_{C,E}(L_j \otimes I) x \text{ for } j = 1,\dots,d. \end{aligned}$$

Let $h \in \mathscr{H}_A$. Because $\tilde{V}_j^E W = W V_j^E$ it follows for all $y \in \Gamma \otimes \mathscr{D}_E$ that

$$\begin{aligned} \langle((L_j \otimes I) \oplus Y_j) W h, W y\rangle &= \langle W(A_j h) \oplus W(e_\emptyset \otimes (D_E)_j h), W y\rangle \\ &= \langle W(e_\emptyset \otimes (D_E)_j h), W y\rangle \\ &= \langle e_\emptyset \otimes (D_E)_j h,\ y\rangle. \end{aligned} \tag{8}$$

Using equations (5) and (7) we write $Wh = u \oplus \lim_{n\to\infty} P_{\mathscr{K}} W v_n$ for some $u \in \Gamma \otimes \mathscr{D}_C, v_n \in \Gamma \otimes \mathscr{D}_E$. Observe that

$$u \oplus \lim_{n\to\infty} \Delta_{C,E} v_n \perp M_{C,E}x \oplus \Delta_{C,E}x \text{ for all } x \in \Gamma \otimes \mathscr{D}_E.$$

Therefore for $j = 1, \dots, d$

$$\begin{aligned}
\langle((L_j \otimes I) \oplus Y_j)Wh, Wy\rangle &= \langle W^*((L_j \otimes I) \oplus Y_j)(u \oplus \lim_{n\to\infty} P_{\mathscr{K}} W v_n),\ y\rangle \\
&= \langle W^*(L_j \otimes I)u,\ y\rangle + \langle W^*(Y_j \lim_{n\to\infty} P_{\mathscr{K}} W v_n),\ y\rangle \\
&= \langle (L_j \otimes I)u, P_{\Gamma\otimes\mathscr{D}_C} Wy\rangle + \langle W^*(\lim_{n\to\infty} \Phi^*_{\mathscr{K}} \Delta_{C,E}(L_j \otimes I)v_n),\ y\rangle \\
&= \langle M^*_{C,E}(L_j \otimes I)u,\ y\rangle + \langle(\lim_{n\to\infty} \Delta_{C,E}(L_j \otimes I)v_n),\ \Phi_{\mathscr{K}} P_{\mathscr{K}} Wy\rangle \\
&= \langle M^*_{C,E}(L_j \otimes I)u,\ y\rangle + \langle(\lim_{n\to\infty} \Delta_{C,E}(L_j \otimes I)v_n), \Delta_{C,E} y\rangle \\
&= \langle M^*_{C,E}(L_j \otimes I)u + \lim_{n\to\infty} \Delta^2_{C,E}(L_j \otimes I)v_n,\ y\rangle. \qquad (9)
\end{aligned}$$

From equations (8) and (9) we have

$$e_\emptyset \otimes (D_E)_j h = M^*_{C,E}(L_j \otimes I)u + \lim_{n\to\infty} \Delta^2_{C,E}(L_j \otimes I)v_n \text{ for } j = 1, \dots, d.$$

By above arguments we conclude that $e_\emptyset \otimes \mathscr{D}_E \subset \mathscr{N}$. For the reverse inclusion we recall from the remark made at the beginning of the proof that

$$M^*_{C,E}(e_\emptyset \otimes \mathscr{D}_C) \subset e_\emptyset \otimes \mathscr{D}_E.$$

Let $u \oplus \lim_{n\to\infty} \Delta_{C,E} v_n \perp M_{C,E}x \oplus \Delta_{C,E}x$ for all $x \in \Gamma \otimes \mathscr{D}_E$ where $u \in \Gamma \otimes \mathscr{D}_C, v_n \in \Gamma \otimes \mathscr{D}_E$. For $i, j \in \{1, \dots, d\}$

$$\begin{aligned}
&(L_i^* \otimes I)[M^*_{C,E}(L_j \otimes I)u + \lim_{n\to\infty} \Delta^2_{C,E}(L_j \otimes I)v_n] \\
&= M^*_{C,E}(L_i^* \otimes I)(L_j \otimes I)u + \lim_{n\to\infty}(L_i^* \otimes I)(I - M^*_{C,E}M_{C,E})(L_j \otimes I)v_n \\
&= \delta_{ij}[M^*_{C,E}u + \lim_{n\to\infty} \Delta^2_{C,E} v_n] = 0.
\end{aligned}$$

where the last equality follows from the fact $u \oplus \lim_{n\to\infty} \Delta_{C,E} v_n \perp M_{C,E}x \oplus \Delta_{C,E}x$ for all $x \in \Gamma \otimes \mathscr{D}_E$. Hence $\mathscr{N} \subset e_\emptyset \otimes \mathscr{D}_E$. $\square$

PROPOSITION 2. *The symbol $\Theta_{C,E} = M_{C,E}|_{e_\emptyset \otimes \mathscr{D}_E}$ of the characteristic function $M_{C,E}$ is an injective map.*

Proof. Let $d_E \in \mathscr{D}_E$ such that $\Theta_{C,E}(e_\emptyset \otimes d_E) = 0$. For $d_C \in \mathscr{D}_C$ we obtain

$$\langle e_\emptyset \otimes d_E, M^*_{C,E}(e_\emptyset \otimes d_C)\rangle = \langle M_{C,E}(e_\emptyset \otimes d_E), e_\emptyset \otimes d_C\rangle = 0, \text{ i.e.,}$$

$$e_\emptyset \otimes d_E \perp M^*_{C,E}(e_\emptyset \otimes \mathscr{D}_C). \qquad (10)$$

Let $u \in \Gamma \otimes \mathscr{D}_C$ and $v_n \in \Gamma \otimes \mathscr{D}_E$ be such that $(u \oplus \lim_{n\to\infty} \Delta_{C,E} v_n) \perp M_{C,E} x \oplus \Delta_{C,E} x$ for all $x \in \Gamma \otimes \mathscr{D}_E$. Then for $j = 1, \dots, d$

$$\begin{aligned}
&\langle e_\emptyset \otimes d_E, M_{C,E}^*(L_j \otimes I)u + \lim_{n\to\infty} \Delta_{C,E}^2 (L_j \otimes I) v_n \rangle \\
&= \langle e_\emptyset \otimes d_E, M_{C,E}^*(L_j \otimes I)u \rangle + \langle e_\emptyset \otimes d_E, \lim_{n\to\infty} \Delta_{C,E}^2 (L_j \otimes I) v_n \rangle \\
&= \langle M_{C,E}(e_\emptyset \otimes d_E), (L_j \otimes I)u \rangle + \lim_{n\to\infty} \langle \Delta_{C,E}^2 (e_\emptyset \otimes d_E), (L_j \otimes I) v_n \rangle \\
&= 0 + \lim_{n\to\infty} \langle (I - M_{C,E}^* M_{C,E})(e_\emptyset \otimes d_E), (L_j \otimes I) v_n \rangle \\
&= \lim_{n\to\infty} \langle e_\emptyset \otimes d_E, (L_j \otimes I) v_n \rangle \\
&= \lim_{n\to\infty} \langle (L_j^* \otimes I)(e_\emptyset \otimes d_E), v_n \rangle = 0.
\end{aligned}$$

Hence using equation (10) and Proposition 1 we deduce that $d_E = 0$. □

3. Characteristic functions of associated liftings

DEFINITION 3. If M and M' are multi-analytic operators with symbols $\Theta : \mathscr{D} \to \Gamma \otimes \mathscr{L}$ and $\Theta' : \mathscr{D}' \to \Gamma \otimes \mathscr{L}$ (with the same $\mathscr{L}$) and there exists a unitary $v : \mathscr{D} \to \mathscr{D}'$ such that $\Theta' \circ v = \Theta$, then we say that M and M' are equivalent.

The injectivity property of symbols of characteristic functions of liftings is not only a necessary condition for minimal contractive liftings as seen in Proposition 2 in the previous section but also a sufficient condition in the following sense:

THEOREM 2. *Let $\underline{C} = (C_1, \dots, C_d)$ be a row contraction on a Hilbert space $\mathscr{H}_C$ and $\tilde{M} : \Gamma \otimes \mathscr{D} \to \Gamma \otimes \mathscr{D}_C$ be a contractive multi-analytic operator such that its symbol $\tilde{\Theta} := \tilde{M}|_{e_\emptyset \otimes \mathscr{D}}$ is an injective map. Then there exists a minimal contractive lifting $\underline{E}$ of $\underline{C}$ such that $M_{C,E}$ and $\tilde{M}$ are equivalent.*

Proof. We define operators $\tilde{\Delta} : \Gamma \otimes \mathscr{D} \to \Gamma \otimes \mathscr{D}$ and $W : \Gamma \otimes \mathscr{D} \to (\Gamma \otimes \mathscr{D}_C) \oplus \overline{\tilde{\Delta}(\Gamma \otimes \mathscr{D})}$ by

$$\tilde{\Delta} := (I - \tilde{M}^* \tilde{M})^{1/2} \text{ and } W(x) = \tilde{M} x \oplus \tilde{\Delta} x.$$

Let $\tilde{\mathscr{H}}$ denote the Hilbert space $\mathscr{H}_C \oplus (\Gamma \otimes \mathscr{D}_C) \oplus \overline{\tilde{\Delta}(\Gamma \otimes \mathscr{D})}$. Clearly W is an isometry. By adding to the domain of W, a copy $\mathscr{H}_A$ of the orthogonal complement of $W(\Gamma \otimes \mathscr{D})$ in $(\Gamma \otimes \mathscr{D}_C) \oplus \overline{\tilde{\Delta}(\Gamma \otimes \mathscr{D})}$, we extend W to a unitary

$$W : \mathscr{H}_A \oplus (\Gamma \otimes \mathscr{D}) \to (\Gamma \otimes \mathscr{D}_C) \oplus \overline{\tilde{\Delta}(\Gamma \otimes \mathscr{D})}.$$

Let $\underline{V}^C = (V_1^C, \dots, V_d^C)$ be the minimal isometric dilation of $\underline{C}$ on $\mathscr{H}_C \oplus (\Gamma \otimes \mathscr{D}_C)$. Define $\underline{\tilde{V}} = (\tilde{V}_1, \dots, \tilde{V}_d)$ on $\tilde{\mathscr{H}}$ by $\tilde{V}_j := V_j^C \oplus Y_j$ for $j = 1, \dots, d$ where Y_j is given by

$$Y_j \tilde{\Delta} x = \tilde{\Delta}(L_j \otimes I) x \text{ for } x \in \Gamma \otimes \mathscr{D}.$$

It is easy to see that $\underline{Y} = (Y_1,\dots,Y_d)$ is a row isometry. Because for $x \in \Gamma \otimes \mathscr{D}$

$$\tilde{V}_j(\tilde{M}x \oplus \tilde{\Delta}x) = (L_j \otimes I)\tilde{M}x \oplus Y_j\tilde{\Delta}x = \tilde{M}(L_j \otimes I)x \oplus \tilde{\Delta}(L_j \otimes I)x,$$

the space $W(\Gamma \otimes \mathscr{D}) = \{\tilde{M}x \oplus \tilde{\Delta}x : x \in \Gamma \otimes \mathscr{D}\}$ is invariant under $\tilde{V}_j$ for $j = 1,\dots,d$. Thus $\mathscr{H}_C \oplus \mathscr{H}_A$ is invariant under $\tilde{V}_j^*$ for $j = 1,\dots,d$ and the contraction

$$E_j^* := \tilde{V}_j^*|_{\mathscr{H}_C \oplus \mathscr{H}_A} \text{ for } j = 1,\dots,d.$$

is called *the lifting associated to the multi-analytic operator* $\tilde{M}$. It was proved in Proposition 3.8 of [3] that this lifting is a reduced lifting and hence a minimal contractive lifting.

We write E_j in the block matrix form as

$$E_j = \begin{pmatrix} C_j & 0 \\ B_j & A_j \end{pmatrix}$$

for $j = 1,\dots,d$ with respect to the decomposition $\mathscr{H}_E = \mathscr{H}_C \oplus \mathscr{H}_A$. Let $h_C, h_C' \in \mathscr{H}_C$ and $u \oplus \lim\limits_{n\to\infty} \tilde{\Delta}v_n, u' \oplus \lim\limits_{n\to\infty} \tilde{\Delta}v_n' \in \mathscr{H}_A$ where $u = \sum_{\alpha\in\tilde{\Lambda}} e_\alpha \otimes u_\alpha, u' = \sum_{\alpha\in\tilde{\Lambda}} e_\alpha \otimes u_\alpha'; u_\alpha, u_\alpha' \in \mathscr{D}_C$, and $v_n, v_n' \in \Gamma \otimes \mathscr{D}$. Then

$$\begin{aligned}
&\langle E_j^*(h_C \oplus u \oplus \lim_{n\to\infty} \tilde{\Delta}v_n),\ h_C' \oplus u' \oplus \lim_{n\to\infty} \tilde{\Delta}v_n'\rangle \\
&= \langle \tilde{V}_j^*(h_C \oplus u \oplus \lim_{n\to\infty} \tilde{\Delta}v_n),\ h_C' \oplus u' \oplus \lim_{n\to\infty} \tilde{\Delta}v_n'\rangle \\
&= \langle h_C \oplus u \oplus \lim_{n\to\infty} \tilde{\Delta}v_n,\ \tilde{V}_j(h_C' \oplus u' \oplus \lim_{n\to\infty} \tilde{\Delta}v_n')\rangle \\
&= \langle h_C \oplus u \oplus \lim_{n\to\infty} \tilde{\Delta}v_n,\ V_j^C(h_C' \oplus u') \oplus \lim_{n\to\infty} Y_j\tilde{\Delta}v_n'\rangle \\
&= \langle h_C \oplus u \oplus \lim_{n\to\infty} \tilde{\Delta}v_n,\ (C_jh_C' \oplus e_\emptyset \otimes (D_C)_jh_C' \oplus (L_j \otimes I)u') \oplus \lim_{n\to\infty} Y_j\tilde{\Delta}v_n'\rangle \\
&= \langle (C_j^*h_C + (D_C)_j^*u_\emptyset) \oplus (L_j^* \otimes I)u \oplus \lim_{n\to\infty} Y_j^*\tilde{\Delta}v_n,\ h_C' \oplus u' \oplus \lim_{n\to\infty} \tilde{\Delta}v_n'\rangle.
\end{aligned}$$

We conclude that

$$\begin{aligned}
B_j^*(u \oplus \lim_{n\to\infty} \tilde{\Delta}v_n) &= (D_C)_j^*u_\emptyset, \\
A_j^*(u \oplus \lim_{n\to\infty} \tilde{\Delta}v_n) &= (L_j^* \otimes I)u \oplus \lim_{n\to\infty} Y_j^*\tilde{\Delta}v_n
\end{aligned}$$

for $j = 1,\dots,d$. Because $u' \oplus \lim\limits_{n\to\infty} \tilde{\Delta}v_n' \in \mathscr{H}_A$ and $\mathscr{H}_A \perp W(\Gamma \otimes \mathscr{D})$, we have $\tilde{M}^*u' + \lim\limits_{n\to\infty} \tilde{\Delta}^2 v_n' = 0$. Further

$$\begin{aligned}
&\langle (I - \tilde{M}\tilde{M}^*)(e_\emptyset \otimes (D_C)_jh_C) \oplus (-\tilde{\Delta}\tilde{M}^*(e_\emptyset \otimes (D_C)_jh_C)),\ u' \oplus \lim_{n\to\infty} \tilde{\Delta}v_n'\rangle \\
&= \langle e_\emptyset \otimes (D_C)_jh_C - \tilde{M}\tilde{M}^*(e_\emptyset \otimes (D_C)_jh_C), u'\rangle - \langle \tilde{M}^*(e_\emptyset \otimes (D_C)_jh_C),\ \lim_{n\to\infty} \tilde{\Delta}^2v_n'\rangle \\
&= \langle e_\emptyset \otimes (D_C)_jh_C,\ u'\rangle - \langle \tilde{M}^*(e_\emptyset \otimes (D_C)_jh_C),\ \tilde{M}^*u'\rangle + \langle \tilde{M}^*(e_\emptyset \otimes (D_C)_jh_C),\ \tilde{M}^*u'\rangle \\
&= \langle (D_C)_jh_C,\ u_\emptyset'\rangle = \langle h_C,\ B_j^*(u' \oplus \lim_{n\to\infty} \tilde{\Delta}v_n')\rangle \\
&= \langle B_jh_C,\ u' \oplus \lim_{n\to\infty} \tilde{\Delta}v_n'\rangle
\end{aligned}$$

Thus, we have

$$B_j h_C = (I - \tilde{M}\tilde{M}^*)(e_\emptyset \otimes (D_C)_j h_C) \oplus (-\tilde{\Delta}\tilde{M}^*(e_\emptyset \otimes (D_C)_j h_C)).$$

Observe that

$$\begin{aligned} A_j(u \oplus \lim_{n\to\infty} \tilde{\Delta} v_n) &= P_{\mathscr{H}_A}\big((L_j \otimes I)u \oplus \lim_{n\to\infty} Y_j \tilde{\Delta} v_n\big) \\ &= ((L_j \otimes I)u \oplus \lim_{n\to\infty} \tilde{\Delta}(L_j \otimes I)v_n) - (\tilde{M}x_j \oplus \tilde{\Delta}x_j) \end{aligned}$$

where each $x_j \in \Gamma \otimes \mathscr{D}$ is defined by

$$\langle (((L_j \otimes I)u - \tilde{M}x_j) \oplus (\lim_{n\to\infty} \tilde{\Delta}(L_j \otimes I)v_n - \tilde{\Delta}x_j),\ (\tilde{M}x \oplus \tilde{\Delta}x)\rangle = 0$$

for $x \in \Gamma \otimes \mathscr{D}$. The above equation implies

$$x_j = \tilde{M}^*(L_j \otimes I)u + \lim_{n\to\infty} \tilde{\Delta}^2 (L_j \otimes I)v_n$$

for $j = 1, \ldots, d$.

Define $\gamma : \mathscr{D}_{*,A} \to \mathscr{D}_C$ by

$$\gamma D_{*,A}(u \oplus \lim_{n\to\infty} \tilde{\Delta} v_n) = u_\emptyset$$

for $u \oplus \lim_{n\to\infty} \tilde{\Delta} v_n \in \mathscr{H}_A$. It is not difficult to check that $B_j^* = (D_C)_j^* \gamma D_{*,A}$ for $j = 1, \ldots, d$.

Next we prove the following three identities for $i, j \in \{1, \ldots, d\}$:

(I1) $\langle (D_E)_j h_C, (D_E)_i h'_C\rangle = \langle \tilde{M}^*(e_\emptyset \otimes (D_C)_j h_C), \tilde{M}^*(e_\emptyset \otimes (D_C)_i h'_C)\rangle$ for $h_C, h'_C \in \mathscr{H}_C$.

(I2) $\langle (D_E)_j h_C, (D_E)_i (u \oplus \lim_{n\to\infty} \Delta v_n)\rangle = \langle \tilde{M}^*(e_\emptyset \otimes (D_C)_j h_C), x_i\rangle$ for $h_C \in \mathscr{H}_C, u \oplus \lim_{n\to\infty} \Delta v_n \in \mathscr{H}_A$ and $x_i = \tilde{M}^*(L_i \otimes I)u + \lim_{n\to\infty} \tilde{\Delta}^2 (L_i \otimes I)v_n$.

(I3) $\langle (D_E)_j (u \oplus \lim_{n\to\infty} \Delta v_n), (D_E)_i (u' \oplus \lim_{n\to\infty} \Delta v'_n)\rangle = \langle x_j, x'_i\rangle$ for $u \oplus \lim_{n\to\infty} \Delta v_n, u' \oplus \lim_{n\to\infty} \Delta v'_n \in \mathscr{H}_A$ and $x'_i = \tilde{M}^*(L_i \otimes I)u' + \lim_{n\to\infty} \tilde{\Delta}^2 (L_i \otimes I)v'_n$.

Proof of I1:

$$\begin{aligned} &\langle (D_E)_j h_C, (D_E)_i h'_C\rangle = \langle (D_E)_i^*(D_E)_j h_C, h'_C\rangle = \langle (\delta_{ij} I - E_i^* E_j)h_C, h'_C\rangle \\ &= \langle \delta_{ij} I h_C, h'_C\rangle - \langle E_j h_C, E_j h'_C\rangle = \langle \delta_{ij} I h_C, h'_C\rangle - \langle C_j h_C \oplus B_j h_C, C_j h'_C \oplus B_j h'_C\rangle \\ &= \langle \delta_{ij} I h_C, h'_C\rangle - \langle C_i^* C_j h_C, h'_C\rangle - \langle B_i^* B_j h_C, h'_C\rangle \\ &= \langle (\delta_{ij} I - C_i^* C_j)h_C, h'_C\rangle - \langle (D_C)_i^* \gamma D_{*,A} B_j h_C, h'_C\rangle \\ &= \langle (D_C)_i^*(D_C)_j h_C, h'_C\rangle \\ &\quad - \langle \gamma D_{*,A}\big((I - \tilde{M}\tilde{M}^*)(e_\emptyset \otimes (D_C)_j h_C) \oplus (-\tilde{\Delta}\tilde{M}^*(e_\emptyset \otimes (D_C)_j h_C))\big), (D_C)_i h'_C\rangle \\ &= \langle (D_C)_i^*(D_C)_j h_C, h'_C\rangle - \langle (D_C)_i^*(D_C)_j h_C, h'_C\rangle \\ &\quad + \langle P_{e_\emptyset \otimes \mathscr{D}_C} \tilde{M}\tilde{M}^*(e_\emptyset \otimes (D_C)_j h_C), e_\emptyset \otimes (D_C)_i h'_C\rangle \\ &= \langle \tilde{M}^*(e_\emptyset \otimes (D_C)_j h_C), \tilde{M}^*(e_\emptyset \otimes (D_C)_i h'_C)\rangle. \end{aligned}$$

Proof of I2:

$$\begin{aligned}
&\langle (D_E)_j h_C, (D_E)_i(u \oplus \lim_{n\to\infty} \Delta v_n)\rangle = \langle (D_E)_i^*(D_E)_j h_C, u \oplus \lim_{n\to\infty} \Delta v_n\rangle \\
&= \langle (\delta_{ij} I - E_i^* E_j) h_C, (u \oplus \lim_{n\to\infty} \Delta v_n)\rangle = -\langle E_j h_C, E_i(u \oplus \lim_{n\to\infty} \Delta v_n)\rangle \\
&= -\langle C_j h_C \oplus B_j h_C, A_i(u \oplus \lim_{n\to\infty} \Delta v_n)\rangle = -\langle h_C, B_j^* A_i(u \oplus \lim_{n\to\infty} \Delta v_n)\rangle \\
&= -\langle h_C, (D_C)_j^* \gamma D_{*,A}\big(((L_i \otimes I)u - \tilde{M} x_i) \oplus (\lim_{n\to\infty} \tilde{\Delta}(L_i \otimes I) v_n - \tilde{\Delta} x_i)\big)\rangle \\
&= \langle e_\emptyset \otimes (D_C)_j h_C, P_{e_\emptyset \otimes \mathscr{D}_C} \tilde{M} x_i\rangle = \langle \tilde{M}^*(e_\emptyset \otimes (D_C)_j h_C), x_i\rangle.
\end{aligned}$$

Proof of I3:

$$\begin{aligned}
&\langle (D_E)_j(u \oplus \lim_{n\to\infty} \Delta v_n), (D_E)_i(u' \oplus \lim_{n\to\infty} \Delta v_n')\rangle \\
&= \langle (\delta_{ij} I - E_i^* E_j)(u \oplus \lim_{n\to\infty} \Delta v_n), u' \oplus \lim_{n\to\infty} \Delta v_n'\rangle \\
&= \langle (\delta_{ij} I - A_i^* A_j)(u \oplus \lim_{n\to\infty} \Delta v_n), u' \oplus \lim_{n\to\infty} \Delta v_n'\rangle \\
&= \langle \delta_{ij} I(u \oplus \lim_{n\to\infty} \tilde{\Delta} v_n) - A_i^* A_j(u \oplus \lim_{n\to\infty} \tilde{\Delta} v_n), u' \oplus \lim_{n\to\infty} \Delta v_n'\rangle \\
&= \langle \delta_{ij} I(u \oplus \lim_{n\to\infty} \tilde{\Delta} v_n) - A_i^*\big(((L_j \otimes I)u - \tilde{M} x_j) \oplus (\lim_{n\to\infty} \tilde{\Delta}(L_j \otimes I) v_n - \tilde{\Delta} x_j)\big), u' \oplus \lim_{n\to\infty} \Delta v_n'\rangle \\
&= \langle \delta_{ij} I(u \oplus \lim_{n\to\infty} \tilde{\Delta} v_n) - [((L_i^* L_j \otimes I)u - (L_i^* \otimes I)\tilde{M} x_j) \oplus (Y_i^*(\lim_{n\to\infty} \tilde{\Delta}(L_j \otimes I) v_n - \tilde{\Delta} x_j))], \\
&\quad u' \oplus \lim_{n\to\infty} \Delta v_n'\rangle \\
&= \langle (L_i^* \otimes I)\tilde{M} x_j \oplus Y_i^* \tilde{\Delta} x_j, u' \oplus \lim_{n\to\infty} \Delta v_n'\rangle \\
&= \langle x_j, \tilde{M}^*(L_i \otimes I)u' + \lim_{n\to\infty} \tilde{\Delta}^2 (L_i \otimes I) v_n'\rangle = \langle x_j, x_i'\rangle.
\end{aligned}$$

Using I1, I2 and I3 we conclude that for $i, j \in \{1, \ldots, d\}$

$$\begin{aligned}
&\langle (D_E)_j(h_C \oplus (u \oplus \lim_{n\to\infty} \Delta v_n)), (D_E)_i(h_C' \oplus (u' \oplus \lim_{n\to\infty} \Delta v_n'))\rangle \\
&= \langle \tilde{M}^*(e_\emptyset \otimes (D_C)_j h_C) + x_j, M^*(e_\emptyset \otimes (D_C)_i h_C) + x_i'\rangle. \qquad (11)
\end{aligned}$$

It follows that there exist an isometry $\nu : \mathscr{D}_E \to \mathscr{D}$ defined by

$$\nu(D_E(h_1, \ldots, h_d)) = \tilde{M}^*(e_\emptyset \otimes D_C(h_1^C, \ldots, h_d^C)) + \sum_{j=1}^d x_j$$

where $h_j = h_j^C \oplus (u_j \oplus \lim_{n\to\infty} \Delta v_{n,j}) \in \mathscr{H}_E, h_j^C \in \mathscr{H}_C, u_j \oplus \lim_{n\to\infty} \Delta v_{n,j} \in \mathscr{H}_A$ and $x_j = \tilde{M}^*(L_j \otimes I)u_j + \lim_{n\to\infty} \tilde{\Delta}^2 (L_j \otimes I) v_{n,j}$ for $j = 1, \ldots, d$.

We claim that $\nu : \mathscr{D}_E \to \mathscr{D}$ is surjective. To prove this claim it is enough to show

$$\begin{aligned}
e_\emptyset \otimes \mathscr{D} = \tilde{M}^*(e_\emptyset \otimes \mathscr{D}_C) \vee \{&\tilde{M}^*(L_j \otimes I)u + \lim_{n\to\infty} \tilde{\Delta}^2 (L_j \otimes I) v_n : \\
&u \oplus \lim_{n\to\infty} \tilde{\Delta} v_n \perp \tilde{M} x \oplus \tilde{\Delta} x \text{ for all } x \in \Gamma \otimes \mathscr{D},\ j = 1, \ldots, d\}, \qquad (12)
\end{aligned}$$

because the space on the R.H.S. is already contained in the Range ν. Let $f \in \mathscr{D}$ be such that

$$e_{\emptyset} \otimes f \perp \tilde{M}^*(e_{\emptyset} \otimes \mathscr{D}_C) \vee \{\tilde{M}^*(L_j \otimes I)u + \lim_{n\to\infty} \tilde{\Delta}^2(L_j \otimes I)v_n : u \in \Gamma \otimes \mathscr{D}_C, v_n \in \Gamma \otimes \mathscr{D}, \\ u \oplus \lim_{n\to\infty} \tilde{\Delta}v_n \perp \tilde{M}x \oplus \tilde{\Delta}x \text{ for all } x \in \Gamma \otimes \mathscr{D},\ j = 1,\ldots,d\}.$$

Clearly $e_{\emptyset} \otimes f \perp \tilde{M}^*(e_{\emptyset} \otimes \mathscr{D}_C)$ implies

$$P_{e_{\emptyset}\otimes\mathscr{D}_C}\tilde{M}(e_{\emptyset} \otimes f) = 0. \tag{13}$$

For $u \oplus \lim_{n\to\infty} \tilde{\Delta}v_n \perp \{\tilde{M}x \oplus \tilde{\Delta}x \text{ for all } x \in \Gamma \otimes \mathscr{D}\}$ and $j = 1,\ldots,d$ we also obtain

$$\begin{aligned}
&\langle (L_j^* \otimes I)\tilde{M}(e_{\emptyset} \otimes f) \oplus Y_j^*\tilde{\Delta}(e_{\emptyset} \otimes f),\ u \oplus \lim_{n\to\infty} \tilde{\Delta}v_n\rangle \\
&= \langle (L_j^* \otimes I)\tilde{M}(e_{\emptyset} \otimes f),\ u\rangle + \langle Y_j^*\tilde{\Delta}(e_{\emptyset} \otimes f),\ \lim_{n\to\infty} \tilde{\Delta}v_n\rangle \\
&= \langle e_{\emptyset} \otimes f,\ \tilde{M}^*(L_j \otimes I)u + \lim_{n\to\infty} \tilde{\Delta}Y_j\tilde{\Delta}v_n\rangle \\
&= \langle e_{\emptyset} \otimes f,\ \tilde{M}^*(L_j \otimes I)u + \lim_{n\to\infty} \tilde{\Delta}^2(L_j \otimes I)v_n\rangle = 0.
\end{aligned} \tag{14}$$

But for $j = 1,\ldots,d$ and $x \in \Gamma \otimes \mathscr{D}$ we observe that

$$\begin{aligned}
&\langle (L_j^* \otimes I)\tilde{M}(e_{\emptyset} \otimes f) \oplus Y_j^*\tilde{\Delta}(e_{\emptyset} \otimes f), \tilde{M}x \oplus \tilde{\Delta}x\rangle \\
&= \langle \tilde{M}^*(L_j^* \otimes I)\tilde{M}(e_{\emptyset} \otimes f) + (\tilde{\Delta}Y_j^*)\tilde{\Delta}(e_{\emptyset} \otimes f), x\rangle \\
&= \langle (L_j^* \otimes I)(\tilde{M}^*\tilde{M} + \tilde{\Delta}^2)(e_{\emptyset} \otimes f), x\rangle \\
&= \langle (L_j^* \otimes I)(e_{\emptyset} \otimes f), x\rangle = \langle 0, x\rangle = 0.
\end{aligned} \tag{15}$$

So from equations (14) and (15) we infer that

$$(L_j^* \otimes I)\tilde{M}(e_{\emptyset} \otimes f) \oplus Y_j^*\tilde{\Delta}(e_{\emptyset} \otimes f) = 0$$

for $j = 1,\ldots,d$. Thus $(L_j^* \otimes I)\tilde{M}(e_{\emptyset} \otimes f) = 0$ for $j = 1,\ldots,d$ and this together with equation (13) imply that $\tilde{M}(e_{\emptyset} \otimes f) = 0$, i.e., $\tilde{\Theta}(e_{\emptyset} \otimes f) = 0$. Since $\tilde{\Theta}$ is an injective map, it follows that $f = 0$. So equation (12) holds and ν is surjective. Hence, ν is unitary.

Finally we claim that $\Theta_{C,E} = \tilde{\Theta}\nu$. For $h_C \in \mathscr{H}_C$ and $j = 1,\ldots,d$

$$\begin{aligned}
&e_{\emptyset} \otimes [(D_C)_j h_C - \gamma D_{*,A} B_j h_C] \\
&= e_{\emptyset} \otimes [(D_C)_j h_C - \gamma D_{*,A}\big((I - \tilde{M}\tilde{M}^*)(e_{\emptyset} \otimes (D_C)_j h_C) \oplus (-\tilde{\Delta}\tilde{M}^*(e_{\emptyset} \otimes (D_C)_j h_C))\big)] \\
&= P_{e_{\emptyset}\otimes\mathscr{D}_C}\tilde{M}\tilde{M}^*(e_{\emptyset} \otimes (D_C)_j h_C) = P_{e_{\emptyset}\otimes\mathscr{D}_C}\tilde{\Theta}\nu((D_E)_j h_C)
\end{aligned} \tag{16}$$

where last equality of the above equation array follows from the definition of ν. For $\alpha \in \tilde{\Lambda}$ with $|\alpha| \geqslant 1$

$$\begin{aligned}
&e_{\alpha} \otimes \gamma D_{*,A}(A_{\alpha})^* B_j h_C \\
&= e_{\alpha} \otimes \gamma D_{*,A}(A_{\alpha})^*\big((I - \tilde{M}\tilde{M}^*)(e_{\emptyset} \otimes (D_C)_j h_C) \oplus (-\tilde{\Delta}\tilde{M}^*(e_{\emptyset} \otimes (D_C)_j h_C))\big) \\
&= e_{\alpha} \otimes \gamma D_{*,A}\big((L_{\alpha}^* \otimes I)(I - \tilde{M}\tilde{M}^*)(e_{\emptyset} \otimes (D_C)_j h_C) \oplus (-Y_{\alpha}^*\tilde{\Delta}\tilde{M}^*(e_{\emptyset} \otimes (D_C)_j h_C))\big) \\
&= -P_{e_{\alpha}\otimes\mathscr{D}_C}\tilde{M}\tilde{M}^*(e_{\emptyset} \otimes (D_C)_j h_C) = -P_{e_{\alpha}\otimes\mathscr{D}_C}\tilde{\Theta}\nu((D_E)_j h_C).
\end{aligned} \tag{17}$$

Combining equations (3), (16) and (17) we obtain

$$\Theta_{C,E}((D_E)_j h_C) = \tilde{\Theta}\nu((D_E)_j h_C) \tag{18}$$

for $h_C \in \mathscr{H}_C$ and $j = 1, \ldots, d$.

It follows from the definition of γ that for $j = 1, \ldots, d$

$$\begin{aligned} e_\emptyset \otimes \gamma D_{*,A} A_j (u \oplus \lim_{n\to\infty} \tilde{\Delta} v_n) &= e_\emptyset \otimes \gamma D_{*,A}\big(((L_j \otimes I)u - \tilde{M}x_j) \oplus (\lim_{n\to\infty} \tilde{\Delta}(L_j \otimes I)v_n - \tilde{\Delta}x_j)\big) \\ &= -P_{e_\emptyset \otimes \mathscr{D}_C} \tilde{M} x_j = -P_{e_\emptyset \otimes \mathscr{D}_C} \tilde{\Theta} x_j \\ &= -P_{e_\emptyset \otimes \mathscr{D}_C} \tilde{\Theta}\nu((D_E)_j (u \oplus \lim_{n\to\infty} \tilde{\Delta} v_n)). \end{aligned} \tag{19}$$

Moreover, for $u \oplus \lim_{n\to\infty} \tilde{\Delta} v_n \in \mathscr{H}_A$ and $i, j = 1, \ldots, d$

$$\begin{aligned} &(\delta_{ij} I - A_i^* A_j)(u \oplus \lim_{n\to\infty} \tilde{\Delta} v_n) \\ &= \delta_{ij} I (u \oplus \lim_{n\to\infty} \tilde{\Delta} v_n) - A_i^* A_j (u \oplus \lim_{n\to\infty} \tilde{\Delta} v_n) \\ &= \delta_{ij} I (u \oplus \lim_{n\to\infty} \tilde{\Delta} v_n) - A_i^* \big(((L_j \otimes I)u - \tilde{M}x_j) \oplus (\lim_{n\to\infty} \tilde{\Delta}(L_j \otimes I)v_n - \tilde{\Delta}x_j)\big) \\ &= \delta_{ij} I (u \oplus \lim_{n\to\infty} \tilde{\Delta} v_n) - [((L_i^* L_j \otimes I)u - (L_i^* \otimes I)\tilde{M}x_j) \oplus (Y_i^*(\lim_{n\to\infty} \tilde{\Delta}(L_j \otimes I)v_n - \tilde{\Delta}x_j))] \\ &= (L_i^* \otimes I)\tilde{M}x_j \oplus Y_i^* \tilde{\Delta} x_j \end{aligned}$$

and so using the definition of γ we deduce that for $\alpha \in \tilde{\Lambda}$ and $i, j = 1, \ldots, d$

$$\begin{aligned} &e_i \otimes e_\alpha \otimes \gamma D_{*,A} (A_\alpha)^* (\delta_{ij} I - A_i^* A_j)(u \oplus \lim_{n\to\infty} \tilde{\Delta} v_n) \\ &= e_i \otimes e_\alpha \otimes \gamma D_{*,A} (A_\alpha)^* ((L_i^* \otimes I)\tilde{M}x_j \oplus Y_i^* \tilde{\Delta} x_j) \\ &= e_i \otimes e_\alpha \otimes \gamma D_{*,A} ((L_\alpha^* L_i^* \otimes I)\tilde{M}x_j \oplus Y_\alpha^* Y_i^* \tilde{\Delta} x_j) \\ &= P_{e_i \otimes e_\alpha \otimes \mathscr{D}_C} \tilde{M} x_j = P_{e_i \otimes e_\alpha \otimes \mathscr{D}_C} \tilde{\Theta} x_j \\ &= P_{e_i \otimes e_\alpha \otimes \mathscr{D}_C} \tilde{\Theta}\nu((D_E)_j (u \oplus \lim_{n\to\infty} \tilde{\Delta} v_n)). \end{aligned} \tag{20}$$

Comparing equations (4), (19) and (20) we obtain

$$\Theta_{C,E}((D_E)_j (u \oplus \lim_{n\to\infty} \tilde{\Delta} v_n)) = \tilde{\Theta}\nu((D_E)_j (u \oplus \lim_{n\to\infty} \tilde{\Delta} v_n)) \tag{21}$$

for $u \oplus \lim_{n\to\infty} \tilde{\Delta} v_n \in \mathscr{H}_A$ and $j = 1, \ldots, d$. This proves the claim that $\Theta_{C,E} = \tilde{\Theta}\nu$. □

Combining Proposition 2 and Theorem 2 we obtain the following:

COROLLARY 1. *Suppose* $\underline{C} = (C_1, \ldots, C_d)$ *is a row contraction on a Hilbert space* $\mathscr{H}_C$ *and* $\tilde{M} : \Gamma \otimes \mathscr{D} \to \Gamma \otimes \mathscr{D}_C$ *is a contractive multi-analytic operator. Then* $\tilde{M}$ *is equivalent to the characteristic function for a minimal contractive lifting of* $\underline{C}$ *if and only if the symbol of* $\tilde{M}$ *is an injective map.*

4. Example

We discuss an example where we do constructions from the previous section for an injective Schur class function Θ and verify Theorem 2. In this example, the function Θ which is not an inner function and so the associated lifting obtained here for the given contraction is not subisometric (cf. Definition 1.1 of [3]).

Let $C = \frac{1}{2}$ be a contraction on a Hilbert space $\mathscr{H}_C = \mathbb{C}$. Assume that $\mathscr{D} = \mathbb{C}$. Set $\Theta(z) = \frac{z}{2}$ and thus $M_\Theta : \mathscr{H}^2 \to \mathscr{H}^2$ is given by $M_\Theta f = \frac{zf}{2} = \frac{M_z f}{2}$ where $M_z : \mathscr{H}^2 \to \mathscr{H}^2$ is the shift operator. Note that this is the same setting as Example 5.3 of [4] and we illustrate the new constructions of Section 3 for this example. As noted there, $\mathscr{H}_A = \left\{ g \oplus \left(-\frac{M_z^* g}{\sqrt{3}} \right) : g \in \mathscr{H}^2 \right\}$ and we obtain the lifting E of C where the operators $B : \mathscr{H}_C \to \mathscr{H}_A$ and $A : \mathscr{H}_A \to \mathscr{H}_A$ are given by

$$B\alpha = \frac{\sqrt{3}}{2}\alpha \oplus \left(-\frac{M_z^*\left(\frac{\sqrt{3}}{2}\alpha\right)}{\sqrt{3}} \right) = \frac{\sqrt{3}}{2}\alpha \oplus 0;$$

$$A\left(g \oplus \left(-\frac{M_z^* g}{\sqrt{3}} \right) \right) = g_A \oplus \left(-\frac{M_z^* g_A}{\sqrt{3}} \right)$$

where $\alpha \in \mathscr{H}_C, g \oplus \left(-\frac{M_z^* g}{\sqrt{3}} \right) \in \mathscr{H}_A$ with $g(z) = \sum_{n=0}^{\infty} g_n z^n$ and $g_A = \frac{3}{4} g_0 z + \sum_{n=2}^{\infty} g_{n-1} z^n$. Since

$$\lim_{n\to\infty} \left\| (A^*)^n \left(g \oplus \left(-\frac{M_z^* g}{\sqrt{3}} \right) \right) \right\|^2 = 0,$$

A is a pure operator and hence it is c.n.c.

For $g \oplus \left(-\frac{M_z^* g}{\sqrt{3}} \right) \in \mathscr{H}_A$ with $g(z) = \sum_{n=0}^{\infty} g_n z^n$

$$\begin{aligned} D_{*,A}^2 \left(g \oplus \left(-\frac{M_z^* g}{\sqrt{3}} \right) \right) &= (I - AA^*) \left(g \oplus \left(-\frac{M_z^* g}{\sqrt{3}} \right) \right) \\ &= \left(g_0 + \frac{g_1}{4} z \right) \oplus \left(-\frac{M_z^* \left(g_0 + \frac{g_1}{4} z \right)}{\sqrt{3}} \right). \end{aligned}$$

The defect operator $D_{*,A} : \mathscr{H}_A \to \mathscr{H}_A$ is given by

$$D_{*,A} \left(g \oplus \left(-\frac{M_z^* g}{\sqrt{3}} \right) \right) = (g_0 + \frac{g_1}{2} z) \oplus \left(-\frac{M_z^* \left(g_0 + \frac{g_1}{2} z \right)}{\sqrt{3}} \right).$$

Because

$$D_A^2\Big(g\oplus\Big(-\frac{M_z^*g}{\sqrt{3}}\Big)\Big)=(I-A^*A)\Big(g\oplus\Big(-\frac{M_z^*g}{\sqrt{3}}\Big)\Big)$$
$$=\frac{g_0}{4}\oplus\Big(-\frac{M_z^*g_0}{\sqrt{3}}\Big)=\frac{g_0}{4}\oplus 0,$$

the defect operator $D_A:\mathscr{H}_A\to\mathscr{H}_A$ is given by $D_A\Big(g\oplus\Big(-\frac{M_z^*g}{\sqrt{3}}\Big)\Big)=\frac{g_0}{2}\oplus 0$. Define a contraction $\gamma:\mathscr{D}_{*,A}\to\mathscr{D}_C$ by

$$\gamma D_{*,A}\Big(g\oplus\Big(-\frac{M_z^*g}{\sqrt{3}}\Big)\Big)=g_0. \tag{22}$$

Observe that $B^*=D_C\gamma D_{*,A}$. We show that γ is resolving. Let $g\oplus\Big(-\frac{M_z^*g}{\sqrt{3}}\Big)\in\mathscr{H}_A$ such that $\gamma D_{*,A}(A^*)^n\Big(g\oplus\Big(-\frac{M_z^*g}{\sqrt{3}}\Big)\Big)=0$ for $n\geqslant 0$. Using the formula of A^* and equation (22), we conclude that $g=0$. Thus we have

$$\gamma D_{*,A}(A^*)^n\Big(g\oplus\Big(-\frac{M_z^*g}{\sqrt{3}}\Big)\Big)=0 \text{ for } n\geqslant 0\Rightarrow D_{*,A}(A^*)^n\Big(g\oplus\Big(-\frac{M_z^*g}{\sqrt{3}}\Big)\Big)=0 \text{ for } n\geqslant 0,$$

i.e., γ is resolving. So by Theorem 1 it follows that E is a minimal contractive lifting of C. For $\alpha\in\mathscr{H}_C$ we have

$$\|D_E\alpha\|^2=|\alpha|^2-\|E\alpha\|^2=|\alpha|^2-\|C\alpha\oplus B\alpha\|^2$$
$$=|\alpha|^2-[\|C\alpha\|^2+\|B\alpha\|^2]=|\alpha|^2-\Big[\frac{|\alpha|^2}{4}+\frac{3|\alpha|^2}{4}\Big]=0.$$

Therefore $\mathscr{D}_E=\overline{\operatorname{Span}}\{D_E\Big(g\oplus\Big(-\frac{M_z^*g}{\sqrt{3}}\Big)\Big):g\oplus\Big(-\frac{M_z^*g}{\sqrt{3}}\Big)\in\mathscr{H}_A\}$. It is easy to check that $D_E\Big(g\oplus\Big(-\frac{M_z^*g}{\sqrt{3}}\Big)\Big)=D_A\Big(g\oplus\Big(-\frac{M_z^*g}{\sqrt{3}}\Big)\Big)=\frac{g_0}{2}$ which implies $\operatorname{Span}\Big\{D_E\Big(g\oplus\Big(-\frac{M_z^*g}{\sqrt{3}}\Big)\Big):g\oplus\Big(-\frac{M_z^*g}{\sqrt{3}}\Big)\in\mathscr{H}_A\Big\}$ is a one dimensional closed vector subspace. So $\mathscr{D}_E=\operatorname{Span}\Big\{D_E\Big(g\oplus\Big(-\frac{M_z^*g}{\sqrt{3}}\Big)\Big):g\oplus\Big(-\frac{M_z^*g}{\sqrt{3}}\Big)\in\mathscr{H}_A\Big\}$. Define the unitary operator $\nu:\mathscr{D}_E\to\mathscr{D}$ by the relation

$$\nu\Big(D_E\Big(g\oplus\Big(-\frac{M_z^*g}{\sqrt{3}}\Big)\Big)\Big)=\frac{g_0}{2}$$

for $g\oplus\Big(-\frac{M_z^*g}{\sqrt{3}}\Big)\in\mathscr{H}_A$ with $g(z)=\sum_{n=0}^{\infty}g_nz^n$.

For $g \oplus \Big(-\frac{M_z^* g}{\sqrt{3}}\Big) \in \mathscr{H}_A$ with $g(z) = \sum_{n=0}^{\infty} g_n z^n$ and $g_A = \frac{3}{4} g_0 z + \sum_{n=2}^{\infty} g_{n-1} z^n$ we have

$$\begin{aligned}
&\Theta_{C,E}(z)\Big(D_E\Big(g \oplus \Big(-\frac{M_z^* g}{\sqrt{3}}\Big)\Big)\Big) \\
&= -\gamma D_{*,A} A\Big(g \oplus \Big(-\frac{M_z^* g}{\sqrt{3}}\Big)\Big) + \sum_{n=1}^{\infty} \gamma D_{*,A}(A^*)^{n-1}(I - A^*A)\Big(g \oplus \Big(-\frac{M_z^* g}{\sqrt{3}}\Big)\Big) z^n \\
&= -\gamma D_{*,A}\Big(g_A \oplus \Big(-\frac{M_z^* g_A}{\sqrt{3}}\Big)\Big) + \gamma D_{*,A}(I - A^*A)\Big(g \oplus \Big(-\frac{M_z^* g}{\sqrt{3}}\Big)\Big) z \\
&\quad + \sum_{n=2}^{\infty} \gamma D_{*,A}(A^*)^{n-1}(I - A^*A)\Big(g \oplus \Big(-\frac{M_z^* g}{\sqrt{3}}\Big)\Big) z^n \\
&= 0 + \frac{g_0}{4} z + 0 = \Theta(z) v\Big(D_E\Big(g \oplus \Big(-\frac{M_z^* g}{\sqrt{3}}\Big)\Big)\Big).
\end{aligned}$$

Thus $\Theta_{C,E}(z) = \Theta(z)v$. Hence the symbol $\Theta_{C,E}$ is equivalent to Θ.

5. Series representation for characteristic function

We first address the natural question as to when the characteristic function of lifting is a polynomial.

DEFINITION 4. A row contraction $\underline{N} = (N_1, \ldots, N_d)$ on a Hilbert space $\mathscr{H}$ is called *nilpotent* if there is an element $n \in \{0, 1, 2, \ldots\}$ such that $N_\alpha = 0$ for all $\alpha \in \tilde{\Lambda}$ with $|\alpha| = n$. The *order* of a nilpotent d-tuple $\underline{N}$ is the smallest $n \in \{0, 1, 2, \ldots\}$ with the above mentioned property.

Recall that a pure row isometry is called a *row shift*.

THEOREM 3. *Let $\underline{C} = (C_1, \ldots, C_d)$ be a row contraction on a Hilbert space $\mathscr{H}_C$ and $\underline{E} = (E_1, \ldots, E_d)$ be a minimal contractive lifting of $\underline{C}$ on a Hilbert space $\mathscr{H}_E \supset \mathscr{H}_C$. If the symbol $\Theta_{C,E}$ for the characteristic function of the lifting $\underline{E}$ of $\underline{C}$ is a polynomial of degree $\leqslant n$ where $n \in \{0, 1, 2, \ldots\}$, then there exist subspaces $\mathscr{H}_v$ and $\mathscr{H}_{nil}$ of $\mathscr{H}_A := \mathscr{H}_E \ominus \mathscr{H}_C$ such that $\mathscr{H}_A = \mathscr{H}_v \oplus \mathscr{H}_{nil}$ and each $A_j = P_{\mathscr{H}_A} E_j|_{\mathscr{H}_A}$ has the block matrix representation*

$$A_j = \begin{pmatrix} S_j & * \\ 0 & N_j \end{pmatrix}, \quad i = 1, \cdots, d$$

where $\underline{S} = (S_1, \ldots, S_d)$ is a row shift on $\mathscr{H}_v$ and $\underline{N} = (N_1, \ldots, N_d)$ is a nilpotent row contraction on $\mathscr{H}_{nil}$ of order $\leqslant n$. In particular, if $n = 0$, then $\mathscr{H}_{nil} = \{0\}$ and $A_j = S_j$, $i = 1, \cdots, d$.

Proof. Assume that the symbol $\Theta_{C,E}$ for the characteristic function of the minimal contractive lifting $\underline{E}$ of $\underline{C}$ is a polynomial of degree $\leqslant n$ where $n \in \{0, 1, 2, \ldots\}$.

Therefore, the series representation given by equation (4) for $\Theta_{C,E}(D_E)_j h_A$ is also a polynomial of degree $\leqslant n$ where $h_A \in \mathscr{H}_A, j = 1,\ldots,d$. Then

$$\gamma D_{*,A}(A_\alpha)^* P_i D_A = 0, \text{ for } |\alpha| \geqslant n \geqslant 0,\ i = 1,\ldots,d$$

where P_i denotes the orthogonal protection of $\oplus_1^d \mathscr{H}_A$ onto the i-the component of $\oplus_1^d \mathscr{H}_A$. Since $\underline{E}$ is the minimal contractive lifting of $\underline{C}$, the contraction $\gamma : \mathscr{D}_{*,A} \to \mathscr{D}_C$ is a resolving map and $\underline{A}$ is a c.n.c. tuple. So $D_{*,A}(A_\alpha)^* P_i D_A = 0$, for $|\alpha| \geqslant n \geqslant 0$, $i = 1,\ldots,d$ which implies that the characteristic function Θ_A for the tuple $\underline{A}$ (cf. [11]) is a polynomial of degree $\leqslant n$. Thus, the theorem follows from Theorem 1.1 of [12] on using the fact that $\underline{A}$ is c.n.c. $\square$

Though the next theorem is not the converse of the previous theorem, it is a result in that direction.

THEOREM 4. *Let $\underline{C} = (C_1,\ldots,C_d)$ be a contraction on a Hilbert space $\mathscr{H}_C$ and $\underline{E} = (E_1,\ldots,E_d)$ be a contractive lifting of $\underline{C}$ on a Hilbert space $\mathscr{H}_E \supset \mathscr{H}_C$. Suppose $\mathscr{H}_A := \mathscr{H}_E \ominus \mathscr{H}_C$ has the decomposition $\mathscr{H}_A = \mathscr{H}_1 \oplus \mathscr{H}_0 \oplus \mathscr{H}_{-1}$ where $\mathscr{H}_1, \mathscr{H}_0$ and $\mathscr{H}_{-1}$ are subspaces of $\mathscr{H}_A$ such that $A_j := P_{\mathscr{H}_A} E_j|_{\mathscr{H}_A}$ has the block matrix representation*

$$A_j = \begin{pmatrix} S_j & * & * \\ 0 & N_j & * \\ 0 & 0 & W_j \end{pmatrix}, \quad j = 1,\ldots,d \tag{23}$$

where $\underline{S} = (S_1,\ldots,S_d)$ is a row shift on $\mathscr{H}_1$, $\underline{N} = (N_1,\ldots,N_d)$ is a nilpotent row contraction on $\mathscr{H}_0$ of order n for some $n \in \{0,1,2,\ldots\}$ and $\underline{W} = (W_1,\ldots,W_d)$ is a coisometry on $\mathscr{H}_{-1}$. Then the symbol $\Theta_{C,E}$ of the characteristic function of the lifting $\underline{E}$ of $\underline{C}$ is a polynomial of degree $\leqslant n$.

Proof. We assume that $\underline{A}$ as defined using the lifting $\underline{E}$ has the form (23). By the proof of the Theorem 1.2 of [12] it follows that

$$D_{*,A}(A_\beta)^* P_i D_A = 0, \text{ for } |\beta| \geqslant n \geqslant 0,\ i = 1,\ldots,d. \tag{24}$$

The symbol $\Theta_{C,E}$ is a polynomial of degree $\leqslant n$ if and only if series representations given by equations (3) and (4) of $\Theta_{C,E}(D_E)_j h_C$ and $\Theta_{C,E}(D_E)_j h_A$, resp. are polynomials of degree $\leqslant n$ where $h_C \in \mathscr{H}_C, h_A \in \mathscr{H}_A, j = 1,\ldots,d$. Using equation (24) it easy to see that the series representation of $\Theta_{C,E}(D_E)_j h_A$ is a polynomial of degree $\leqslant n$ for $h_A \in \mathscr{H}_A, j = 1,\ldots,d$. To prove that the series representation of $\Theta_{C,E}(D_E)_j h_C$ is a polynomial of degree $\leqslant n$ for $h_C \in \mathscr{H}_C, j = 1,\ldots,d$, it is enough to show that

$$\gamma D_{*,A}(A_\alpha)^* B_i = 0 \text{ for } |\alpha| \geqslant n+1,\ i = 1,\ldots,d.$$

Here $B_i^* = (D_C)_i \gamma D_{*,A}$ where $(D_C)_i : \mathscr{H}_C \to \oplus_1^d \mathscr{H}_C$ is defined by

$$(D_C)_i h_C = D_C(0,\ldots,h_C,\ldots,0)$$

with $h_C \in \mathscr{H}_C$ embedded at i^{th} place. Note that

$$A_i^* D_{*,A}^2 = P_i D_A^2 \underline{A}^*, \ \ i = 1, \ldots, d.$$

For any $\alpha \in \tilde{\Lambda}$ with $|\alpha| \geqslant n+1$, there exist $\beta \in \tilde{\Lambda}$ with $|\beta| \geqslant n$ and $\alpha = j\beta$ for some $j \in \{1, \ldots, d\}$. Therefore, from equation (24) we have

$$\begin{aligned} 0 =& D_{*,A}(A_\beta)^* P_j D_A^2 \underline{A}^* = D_{*,A}(A_\beta)^* A_j^* D_{*,A}^2 = D_{*,A}(A_j A_\beta)^* D_{*,A}^2 = \\ =& D_{*,A}(A_\alpha)^* D_{*,A}^2 = D_{*,A}(A_\alpha)^* D_{*,A}. \end{aligned}$$

The last equality in the above equation array follows using the fact that orthogonal complement of $\ker D_{*,A}$ is $\mathscr{D}_{*,A}$. Hence

$$\gamma D_{*,A}(A_\alpha)^* B_j = \gamma D_{*,A}(A_\alpha)^* D_{*,A} \gamma^* (D_C)_j^* = 0.$$

Thus, we have shown that

$$\gamma D_{*,A}(A_\alpha)^* B_i = 0 \text{ for } |\alpha| \geqslant n+1, \ i = 1, \ldots, d. \quad \square$$

Let $\underline{C}$ be a row contraction on a Hilbert space $\mathscr{H}_C$. Let $\Theta_{C,E}$ be the symbol of the characteristic function of a contractive lifting $\underline{E}$ of $\underline{C}$ by $\underline{A}$ on a Hilbert space $\mathscr{H}_E = \mathscr{H}_C \oplus \mathscr{H}_A$. For $i \in \{1, \ldots, d\}$, the element $\Theta_{C,E}(D_E)_i h$ has two different series expansion, namely those given by equations (3) and (4) for $h \in \mathscr{H}_C$ and $h \in \mathscr{H}_A$, resp.. It is always preferable to have a single series expansion for $\Theta_{C,E}(D_E)_i h$ for all $h \in \mathscr{H}_E$ for carrying out further investigation about the properties of $\Theta_{C,E}$. In the next theorem we establish a single series expansion for the characteristic function of the lifting $\underline{E}$.

THEOREM 5. *Let $\underline{E}$ be a contractive lifting of a row contraction $\underline{C}$ by $\underline{A}$ and $\Theta_{C,E}$ denote the symbol of the characteristic function of the lifing $\underline{E}$. Set*

$$\tilde{C} := \sum_{j=1}^{d} (D_C)_j P_{\mathscr{H}_C} E_j^* : \mathscr{H}_E \to \mathscr{D}_C, \ \ \tilde{D} := \sum_{j=1}^{d} (D_C)_j P_{\mathscr{H}_C} (D_E)_j^* : \mathscr{D}_E \to \mathscr{D}_C$$

where $P_{\mathscr{H}_C}$ be the orthogonal projection of $\mathscr{H}_E$ on to $\mathscr{H}_C$. Then for $h \in \mathscr{H}_E$ and $i = 1, \ldots, d$

$$\begin{aligned} (I \otimes D_C^2)\Theta_{C,E}(D_E)_i h = e_\emptyset \otimes \tilde{D}(D_E)_i h + & \sum_{\beta \in \tilde{\Lambda}, j=1,\ldots,d} e_j \otimes e_\beta \\ & \otimes \tilde{C} P_{\mathscr{H}_A} (E_\beta)^* (D_E)_j^* (D_E)_i h \end{aligned} \tag{25}$$

where $P_{\mathscr{H}_A}$ is the orthogonal projection of $\mathscr{H}_E$ on to $\mathscr{H}_A$.

Proof. For $h_A \in \mathscr{H}_A$ we obtain

$$\begin{aligned} \tilde{C} h_A &= \sum_{j=1}^{d} (D_C)_j P_{\mathscr{H}_C} E_j^* h_A = \sum_{j=1}^{d} (D_C)_j P_{\mathscr{H}_C} (B_j^* h_A \oplus A_j^* h_A) \\ &= \sum_{j=1}^{d} (D_C)_j B_j^* h_A = D_C \underline{B}^* h_A = D_C^2 \gamma D_{*,A} h_A. \end{aligned} \tag{26}$$

We consider two cases:

Case 1. For $h_C \in \mathscr{H}_C$ and $i = 1, \ldots, d$

$$
\begin{aligned}
\tilde{D}(D_E)_i h_C &= \sum_{j=1}^{d} (D_C)_j P_{\mathscr{H}_C} (D_E)_j^* (D_E)_i h_C = \sum_{j=1}^{d} (D_C)_j P_{\mathscr{H}_C} (\delta_{ij} I - E_j^* E_i) h_C \\
&= (D_C)_i h_C - \Big(\sum_{j=1}^{d} (D_C)_j P_{\mathscr{H}_C} E_j^* \Big) E_i h_C = (D_C)_i h_C - \tilde{C} E_i h_C \\
&= (D_C)_i h_C - \tilde{C}(C_i h_C \oplus B_i h_C) = (D_C)_i h_C - \tilde{C} C_i h_C - \tilde{C} B_i h_C \\
&= (D_C)_i h_C - (\sum_{j=1}^{d} (D_C)_j P_{\mathscr{H}_C} E_j^*) C_i h_C - \tilde{C} B_i h_C \\
&= (D_C)_i h_C - \sum_{j=1}^{d} (D_C)_j C_j^* C_i h_C - \tilde{C} B_i h_C \\
&= \sum_{j=1}^{d} (D_C)_j (\delta_{ij} h_C - C_j^* C_i h_C) - \tilde{C} B_i h_C
\end{aligned} \tag{27}
$$

Observe that

$$
\begin{aligned}
D_C^2 (D_C)_i h_C &= D_C^2 D_C (0, \ldots, h_C, \ldots, 0) = D_C D_C^2 (0, \ldots, h_C, \ldots, 0) \\
&= D_C (-C_1^* C_i h_C, \ldots, (I - C_i^* C_i) h_C, \ldots, -C_d^* C_i h_C) \\
&= \sum_{j=1}^{d} (D_C)_j (\delta_{ij} h_C - C_j^* C_i h_C)
\end{aligned} \tag{28}
$$

for $i = 1, \ldots, d$. From equations (26), (27) and (28) it follows that

$$
\tilde{D}(D_E)_i h_C = D_C^2 [(D_C)_i h_C - \gamma D_{*,A} B_i h_C] \tag{29}
$$

for $h_C \in \mathscr{H}_C$ and $i = 1, \ldots, d$.

Further, for $i, j = 1, \ldots, d$ and $\beta \in \tilde{\Lambda}$ we have:

$$
\begin{aligned}
&\tilde{C} P_{\mathscr{H}_A} (E_\beta)^* (D_E)_j^* (D_E)_i h_C \\
&= \tilde{C} P_{\mathscr{H}_A} (E_\beta)^* (\delta_{ij} I - E_j^* E_i) h_C = -\tilde{C} P_{\mathscr{H}_A} (E_\beta)^* E_j^* E_i h_C \\
&= -\tilde{C} P_{\mathscr{H}_A} (E_\beta)^* ((C_j^* C_i + B_j^* B_i) h_C \oplus A_j^* B_i h_C) \\
&= -\tilde{C} [P_{\mathscr{H}_A} (E_\beta)^* (C_j^* C_i + B_j^* B_i) h_C + P_{\mathscr{H}_A} (E_\beta)^* A_j^* B_i h_C] \\
&= -\tilde{C} [0 + (A_\beta)^* A_j^* B_i h_C] \\
&= -D_C^2 \gamma D_{*,A} (A_{j\beta})^* B_i h_C
\end{aligned} \tag{30}
$$

where last equality follows from equation (26). Combining the equations (4), (29) and (30) we conclude that for $i = 1,\dots,d$

$$
\begin{aligned}
&(I\otimes D_C^2)\Theta_{C,E}(D_E)_i h_C \\
&= e_\emptyset \otimes D_C^2[(D_C)_i h_C - \gamma D_{*,A}B_i h_C] - \sum_{|\alpha|\geqslant 1} e_\alpha \otimes D_C^2\gamma D_{*,A}(A_\alpha)^* B_i h_C \\
&= e_\emptyset \otimes D_C^2[(D_C)_i h_C - \gamma D_{*,A}B_i h_C] - \sum_{\beta\in\tilde\Lambda, j=1,\dots,d} e_j\otimes e_\beta \otimes D_C^2\gamma D_{*,A}(A_{j\beta})^* B_i h_C \\
&= e_\emptyset \otimes \tilde D(D_E)_i h_C + \sum_{\beta\in\tilde\Lambda, j=1,\dots,d} e_j\otimes e_\beta\otimes \tilde C P_{\mathscr{H}_A}(E_\beta)^*(D_E)_j^*(D_E)_i h_C.
\end{aligned}
$$

Case 2: For $h_A \in \mathscr{H}_A$ and $i = 1,\dots,d$ we have

$$
\begin{aligned}
\tilde D(D_E)_i h_A &= \sum_{j=1}^d (D_C)_j P_{\mathscr{H}_C}(D_E)_j^*(D_E)_i h_A \\
&= \sum_{j=1}^d (D_C)_j P_{\mathscr{H}_C}(\delta_{ij}I - E_j^*E_i)h_A = -\Big(\sum_{j=1}^d (D_C)_j P_{\mathscr{H}_C}E_j^*\Big)E_i h_A \\
&= -\tilde C A_i h_A = -D_C^2\gamma D_{*,A}A_i h_A \qquad (31)
\end{aligned}
$$

where last equality follows from equation (26). Also for $i, j = 1,\dots,d$ and $\beta\in\tilde\Lambda$, we have

$$
\begin{aligned}
\tilde C P_{\mathscr{H}_A}(E_\beta)^*(D_E)_j^*(D_E)_i h_A &= \tilde C P_{\mathscr{H}_A}(E_\beta)^*(\delta_{ij}I - E_j^*E_i)h_A \\
&= \tilde C(\delta_{ij}P_{\mathscr{H}_A}(E_\beta)^*h_A - P_{\mathscr{H}_A}(E_\beta)^*E_j^*E_i h_A) \\
&= \tilde C(\delta_{ij}(A_\beta)^*h_A - (A_\beta)^*A_j^*A_i h_A) \\
&= \tilde C(A_\beta)^*(\delta_{ij}I - A_j^*A_i)h_A \\
&= D_C^2\gamma D_{*,A}(A_\beta)^*(\delta_{ij}I - A_j^*A_i)h_A \qquad (32)
\end{aligned}
$$

where last equality follows from equation (26). Finally, equations (4), (31) and (32) yield for $i = 1,\dots,d$

$$
\begin{aligned}
&(I\otimes D_C^2)\Theta_{C,E}(D_E)_i h_A \\
&= -e_\emptyset\otimes D_C^2\gamma D_{*,A}A_i h_A + \sum_{\beta\in\tilde\Lambda, j=1,\dots,d} e_j\otimes e_\beta\otimes D_C^2\gamma D_{*,A}(A_\beta)^*(\delta_{ij}I - A_j^*A_i)h_A \\
&= e_\emptyset\otimes\tilde D(D_E)_i h_A + \sum_{\beta\in\tilde\Lambda, j=1,\dots,d} e_j\otimes e_\beta\otimes\tilde C P_{\mathscr{H}_A}(E_\beta)^*(D_E)_j^*(D_E)_i h_A,
\end{aligned}
$$

and this proves the theorem. □

As an application of the series representation obtained in Theorem 5 we show that certain expression involving the characteristic function of any contractive lifting can be realised as a transfer function of a linear system. Let $\underline{C}$ be a row contraction on a Hilbert space $\mathscr{H}_C$. Let $\Theta_{C,E}$ be the symbol of the characteristic function of a contractive lifting $\underline{E}$ of $\underline{C}$ by $\underline{A}$ on a Hilbert space $\mathscr{H}_E = \mathscr{H}_C \oplus \mathscr{H}_A$. Let $\tilde{C}$ and $\tilde{D}$ be defined as in Theorem 5. Define a *colligation* of operators (cf. [2]) by

$$\mathscr{C}_{C,E} := \begin{pmatrix} E_1^* & (D_E)_1^* \\ \vdots & \vdots \\ E_d^* & (D_E)_d^* \\ \tilde{C}P_{\mathscr{H}_A} & \tilde{D} \end{pmatrix} : \mathscr{H}_E \oplus \mathscr{D}_E \to \bigoplus_{j=1}^{d} \mathscr{H}_E \oplus \mathscr{D}_C.$$

From the colligation $\mathscr{C}_{C,E}$ we get the following $\tilde{\Lambda}$-*linear system* $\Sigma_{C,E}$:

$$x(j\alpha) = E_j^* x(\alpha) + (D_E)_j^* u(\alpha), \tag{33}$$

$$y(\alpha) = \tilde{C}P_{\mathscr{H}_A} x(\alpha) + \tilde{D}u(\alpha) \tag{34}$$

where $j = 1,\dots,d$ and $\alpha, j\alpha$ are words in $\tilde{\Lambda}$, and

$$x : \tilde{\Lambda} \to \mathscr{H}_E, \quad u : \tilde{\Lambda} \to \mathscr{D}_E, \quad y : \tilde{\Lambda} \to \mathscr{D}_C.$$

This $\tilde{\Lambda}$-linear system is a noncommutative Fornasini-Marchesini system (cf. [1]). In this $\tilde{\Lambda}$-linear system, u takes values in the *input space* $\mathscr{D}_E$ and y takes values in the *output space* $\mathscr{D}_C$. Let $z = (z_1,\dots,z_d)$ be a d-tuple of formal noncommuting indeterminates. Define the Fourier transforms of x, u and y as

$$\hat{x}(z) = \sum_{\alpha\in\tilde{\Lambda}} x(\alpha)z^\alpha, \quad \hat{u}(z) = \sum_{\alpha\in\tilde{\Lambda}} u(\alpha)z^\alpha \text{ and } \hat{y}(z) = \sum_{\alpha\in\tilde{\Lambda}} y(\alpha)z^\alpha$$

respectively where $z^\alpha = z_{\alpha_n}\dots z_{\alpha_1}$ for $\alpha = \alpha_n\dots\alpha_1 \in \tilde{\Lambda}$. Suppose that z-variables commute with the coefficients of equations (33) and (34). Then, the input-output relation

$$\hat{y}(z) = \Upsilon_{C,E}(z)\hat{u}(z)$$

is obtained on assuming $x(\emptyset) := 0$ with

$$\Upsilon_{C,E}(z) := \sum_{\alpha\in\tilde{\Lambda}} \Upsilon_{C,E}^{(\alpha)} z^\alpha := \tilde{D} + \tilde{C}P_{\mathscr{H}_E} \sum_{\beta\in\tilde{\Lambda}, j=1,\dots,d} (E_{\bar{\beta}})^*(D_E)_j^* z^{\beta j}. \tag{35}$$

Here $\bar{\beta} = \beta_1\dots\beta_n$ is the reverse of $\beta = \beta_n\dots\beta_1 \in \tilde{\Lambda}$. The formal noncommutative power series $\Upsilon_{C,E}$ is called the *transfer function* associated to the colligation matrix $\mathscr{C}_{C,E}$.

For $y(\alpha) \in \mathscr{Y}$ with $\sum_{\alpha\in\tilde{\Lambda}} \|y(\alpha)\|^2 < \infty$, the series $\sum_{\alpha\in\tilde{\Lambda}} y(\alpha)z^\alpha$ stands for a series converging to an element of $\ell^2(\tilde{\Lambda}, \mathscr{Y})$. Define unitaries $\tilde{M} : \ell^2(\tilde{\Lambda}, \mathscr{D}_C) \to \mathscr{F} \otimes \mathscr{D}_C$ and $\Phi : (\mathscr{D}_E)z^\emptyset \to e_\emptyset \otimes \mathscr{D}_E$ by

$$\tilde{M}\Big(\sum_{\alpha\in\tilde{\Lambda}} y_\alpha z^\alpha\Big) := \sum_{\alpha\in\tilde{\Lambda}} e_{\bar{\alpha}} \otimes y_\alpha \text{ and } \Phi(uz^\emptyset) := e_\emptyset \otimes u.$$

It follows that

$$\tilde{M}\Upsilon_{C,E}(z) = (I \otimes D_C^2)\Theta_{C,E}\tilde{\Phi}.$$

Hence, $(I \otimes D_C^2)\Theta_{C,E}$ is identifiable with the transfer function $\Upsilon_{C,E}$ which is associated with the colligation matrix $\mathscr{C}_{C,E}$.

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